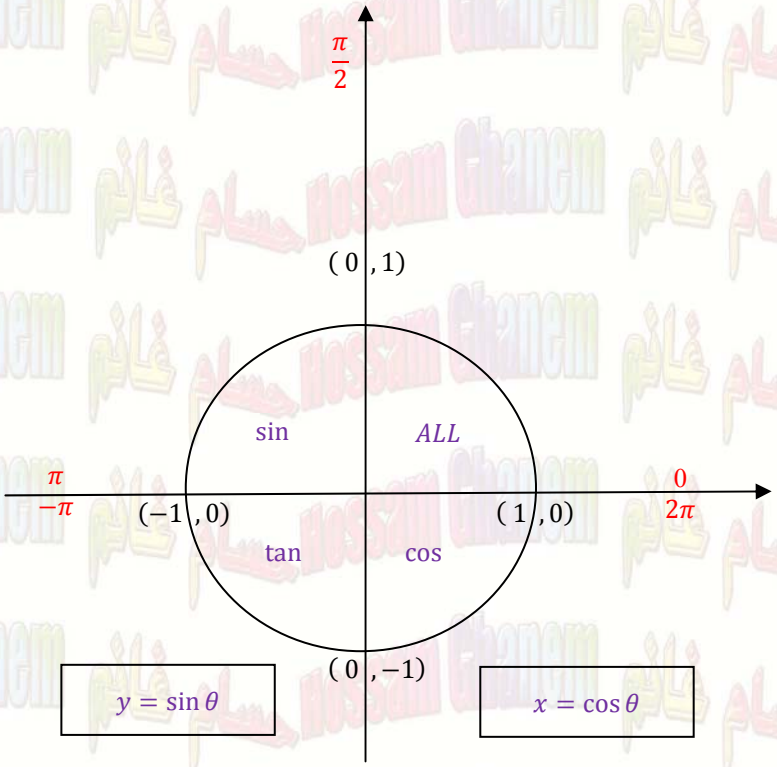
HOSSAM GHANEM

(4) 3.5 *Techniques for finding limits of Trigonometric functions (A)*

Corner Angel

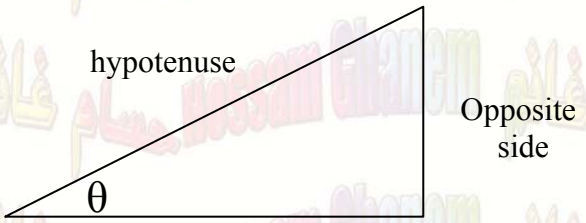
Angel		$\sin x$ y	$\cos x$ x	$\tan x$ y/x
2π	0	0	1	0
$-\frac{3\pi}{2}$	$\frac{\pi}{2}$	1	0	$\pm\infty$
$-\pi$	π	0	-1	0
$-\frac{\pi}{2}$	$\frac{3\pi}{2}$	-1	0	$\pm\infty$



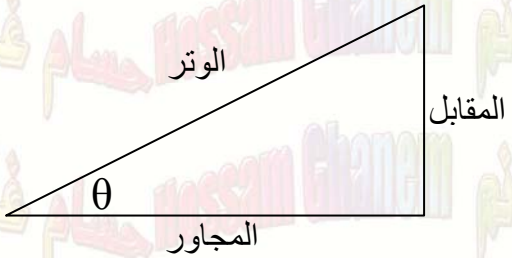
All Student Take Calculus

The Right Triangle

$\sin x$	المقابل / الوتر	$\frac{opp}{hyp}$
$\cos x$	المجاور / الوتر	$\frac{adj}{hyp}$
$\tan x$	المقابل / المجاور	$\frac{opp}{adj}$

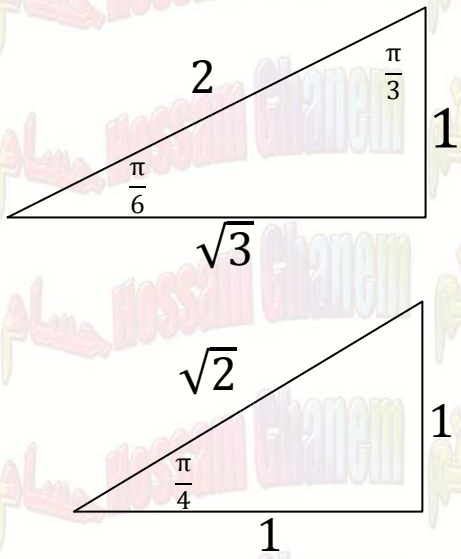


Adiacent side



Special Angles

Angel		$\sin x$	$\cos x$	$\tan x$
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1



Negative Angles

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$

$\sin \leftrightarrow \cos \quad \tan \leftrightarrow \cot$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$
$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$	$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$	$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$

$\sin(\pi - \theta) = \sin\theta$	$\cos(\pi - \theta) = -\cos\theta$	$\tan(\pi - \theta) = -\tan\theta$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\cos(x \pm y) = \cos x \ \cos y \mp \sin x \sin y$

$\sin(2x) = 2\sin x \cos x$	$\cos(2x) = \begin{cases} \cos^2 - \sin^2 x \\ 2\cos^2 - 1 \\ 1 - 2\sin^2 x \end{cases}$

$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$	$1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$
$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1$	$1 + \cos x = 2\cos^2\left(\frac{x}{2}\right)$

Limits

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$\lim_{x\to 1}\frac{\sin(x-1)}{x-1} = 1 \qquad \lim_{x\to \frac{1}{2}}\frac{\sin(2x-1)}{2x-1} \qquad \lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 1$$

$$\lim_{x\to 0}\frac{\tan x}{x} = 1$$

$$\lim_{x\to 1}\frac{\tan(x-1)}{x-1} \qquad \lim_{x\to \frac{1}{2}}\frac{\tan(2x-1)}{2x-1} \qquad \lim_{x\to\infty}\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}} = 1$$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

$\lim_{x\to 1}\frac{1-\cos(x-1)}{x-1}=0$	$\lim_{x\to \frac{1}{2}}\frac{1-\cos(2x-1)}{2x-1}=0$	$\lim_{x\to \infty}\frac{1-\cos\left(\frac{1}{x}\right)}{\frac{1}{x}}==0$

Example 1
31 October 31, 2000

Find the limit , if it exists $\lim_{x\to 0}\frac{1-\cos 2x}{\sin^2 x}$

Solution

$$\lim_{x\to 0}\frac{1-\cos 2x}{\sin^2 x}=\frac{1-1}{0}=\frac{0}{0}$$

$$\lim_{x\to 0}\frac{1-\cos 2x}{\sin^2 x}=\lim_{x\to 0}\frac{2\sin^2 x}{\sin^2 x}=\lim_{x\to 0}2=2$$

Example 2
7 July 29, 1993

Find the limit , if it exists $\lim_{x\to \frac{\pi}{4}}\frac{\cos 2x}{\cos x-\sin x}$

Solution

$$\lim_{x\to \frac{\pi}{4}}\frac{\cos 2x}{\cos x-\sin x}=\frac{0}{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}}=\frac{0}{0}$$

$$\lim_{x\to \frac{\pi}{4}}\frac{\cos 2x}{\cos x-\sin x}=\lim_{x\to \frac{\pi}{4}}\frac{\cos^2 x-\sin^2 x}{\cos x-\sin x}=\lim_{x\to \frac{\pi}{4}}\frac{(\cos x-\sin x)(\cos x+\sin x)}{(\cos x-\sin x)}$$

$$=\lim_{x\to \frac{\pi}{4}}(\cos x+\sin x)=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

Example 3
37 July 12, 2003 A

Find the following limit , if it exists $\lim_{x\to \pi}\cos(x-\sin x)$

Solution

$$\lim_{x\to \pi}\cos(x-\sin x)=\cos(\pi-\sin 0)=\cos(\pi-0)=\cos\pi=-1$$

Example 4

Evaluate the following limit $\lim_{t\to 0}\frac{\sin 3t}{t}$

Solution

$$\lim_{t\to 0}\frac{\sin 3t}{t}=\frac{0}{0}$$

$$\lim_{t\to 0}\frac{\sin 3t}{t}=\lim_{t\to 0}\frac{\sin 3t}{\frac{1}{3}(3t)}=3\lim_{3t\to 0}\frac{\sin 3t}{(3t)}=3(1)=3$$

Example 5
13 February 19, 1995

Find the limit , if it exists $\lim_{x\to 0}\frac{\tan 5x}{\sin 3x}$

Solution

$$\lim_{x\to 0}\frac{\tan 5x}{\sin 3x}=\frac{\tan 0}{\sin 0}=\frac{0}{0}$$

$$\lim_{x\to 0}\frac{\tan 5x}{\sin 3x}=\lim_{x\to 0}\frac{5x\,\frac{\tan 5x}{5x}}{3x\,\frac{\sin 3x}{3x}}=\left(\frac{5}{3}\right)\lim_{x\to 0}\frac{\frac{\tan 5x}{5x}}{\frac{\sin 3x}{3x}}=\frac{5}{3}\cdot\frac{1}{1}=\frac{5}{3}$$

Example 6
37 July 12, 2003 A

Find the limit , if it exists $\lim_{x\to 0}\frac{|x|}{\sin 2x}$

Solution

$$\lim_{x\to 0}\frac{|x|}{\sin 2x}=\frac{0}{0}$$

$$\lim_{x\to 0^+}\frac{|x|}{\sin 2x}=\lim_{x\to 0^+}\frac{x}{\sin 2x}=\lim_{x\to 0^+}\frac{x}{2x\,\frac{\sin 2x}{2x}}=\lim_{x\to 0^+}\frac{1}{2\,\frac{\sin 2x}{2x}}=\frac{1}{2(1)}=\frac{1}{2}$$

$$\lim_{x\to 0^-}\frac{|x|}{\sin 2x}=\lim_{x\to 0^-}\frac{-x}{\sin 2x}=-\frac{1}{2}$$

$$\therefore \lim_{x\to 0}\frac{|x|}{\sin 2x}\quad D.N.E$$

Example 7
42 March 29, 2006

Evaluate the following limit $\lim_{x\to 0}(\sin x)\sqrt{1+\frac{1}{x^2}}$

Solution

$$L=\lim_{x\to 0}(\sin x)\sqrt{1+\frac{1}{x^2}}=0\cdot\infty$$

$$L=\lim_{x\to 0}(\sin x)\sqrt{1+\frac{1}{x^2}}=\lim_{x\to 0}(\sin x)\sqrt{\frac{x^2+1}{x^2}}=\lim_{x\to 0}\frac{\sin x}{|x|}\sqrt{x^2+1}$$

$$\lim_{x\to 0^+}\frac{\sin x}{|x|}\sqrt{x^2+1}=\lim_{x\to 0^+}\frac{\sin x}{x}\sqrt{x^2+1}=(1)\sqrt{0+1}=1$$

$$\lim_{x\to 0^-}\frac{\sin x}{|x|}\sqrt{x^2+1}=\lim_{x\to 0^-}\frac{\sin x}{-x}\sqrt{x^2+1}=-1$$

$$\therefore L\quad D.N.E$$

Example 8
1 January 1990

Find the limit , if it exists $\lim_{x\to 0}\frac{4x^2+3x\sin x}{x^2}$

Solution

$$L=\lim_{x\to 0}\frac{4x^2+3x\sin x}{x^2}=\frac{0}{0}$$

$$L=\lim_{x\to 0}\frac{4x^2+3x\sin x}{x^2}=\lim_{x\to 0}\frac{4x^2}{x^2}+\frac{3x\sin x}{x^2}=\lim_{x\to 0}4+3\cdot\frac{\sin x}{x}=4+3(1)=7$$

Example 9

Find the limit , if it exists $\lim_{x\to 0}\frac{\sin^2 x+\sin 2x}{3x}$

Solution

$$L=\lim_{x\to 0}\frac{\sin^2 x+\sin 2x}{3x}=\frac{0}{0}$$

$$L=\lim_{x\to 0}\frac{\sin^2 x+\sin 2x}{3x}=\lim_{x\to 0}\frac{\sin^2 x+2\sin x\cos x}{3x}$$

$$= \lim_{x\to 0} \frac{\sin x\,(1+2\,\cos x)}{x\cdot 3} = \lim_{x\to 0} \frac{\sin x}{x}\,\frac{(1+2\,\cos x)}{3} = 1\cdot\frac{1+2}{3} = 1$$

Example 10 60 *October* 31 , 2011	(2 points) Find the limit , if it exists . $\lim_{x\to 0}\dfrac{3x+\tan 2x}{2x+3\sin x}$

Solution

$$\lim_{x\to 0}\frac{3x+\tan 2x}{2x+3\sin x} = \frac{0}{0}$$

$$\lim_{x\to 0}\frac{3x+\tan 2x}{2x+3\sin x} = \lim_{x\to 0}\frac{3+\dfrac{\tan 2x}{x}}{2+\dfrac{3\sin x}{x}} = \lim_{x\to 0}\frac{3+2\dfrac{\tan 2x}{2x}}{2+3\dfrac{\sin x}{x}} == \frac{3+2(1)}{2+3(1)} = \frac{5}{5} = 1$$

Example 11 21 March 26, 1998	Evaluate the following limit $\lim_{x\to 0}\left(\dfrac{\tan^3 x}{(x^4+5x^3)}\right)$

Solution

$$L = \lim_{x\to 0}\left(\frac{\tan^3 x}{(x^4+5x^3)}\right) = \frac{0}{0}$$

$$L = \lim_{x\to 0}\left(\frac{\tan^3 x}{(x^4+5x^3)}\right) = \lim_{x\to 0}\left(\frac{\tan^3 x}{x^3(x+5)}\right) = \lim_{x\to 0}\left(\frac{\tan^3 x}{x^3}\right)\cdot\frac{1}{x+5}$$

$$= \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^3\cdot\frac{1}{x+5} = (1)^3\frac{1}{5} = \frac{1}{5}$$

Example 12 24 November 3, 1998	Evaluate the following limit $\lim_{x\to 0}\dfrac{x}{2x\cos x-\sin^2 x}$

Solution

$$L = \lim_{x\to 0}\frac{x}{2x\cos x-\sin^2 x} = \frac{0}{0}$$

$$L = \lim_{x\to 0}\frac{x}{2x\cos x-\sin^2 x} = \lim_{x\to 0}\frac{1}{2\cos x-\dfrac{\sin^2 x}{x}} = \lim_{x\to 0}\frac{1}{2\cos x-\dfrac{\sin x}{x}\cdot\sin x} = \frac{1}{2-1\,(0)} = \frac{1}{2}$$

Example 13 8 August 28, 1993	Evaluate the following limit $\lim_{x\to 0}\dfrac{x(\sin x)}{1-\cos 2x}$

Solution

$$L = \lim_{x\to 0}\frac{x(\sin x)}{1-\cos 2x} = \frac{0}{1-1} = \frac{0}{0}$$

$$L = \lim_{x\to 0}\frac{x(\sin x)}{1-\cos 2x} = \lim_{x\to 0}\frac{x(\sin x)}{2\,\sin^2 x} = \frac{1}{2}\lim_{x\to 0}\frac{x}{\sin x} = \frac{1}{2}(1) = \frac{1}{2}$$

Homework

1	Find the limit , if it exists	$\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{\cos 2x}$	
2	Evaluate the following limit (if it exists)	$\lim_{x\to 0} \frac{\sin x}{\sqrt{x^3 + x^2}}$	43 June 28, 2008
3	Find	$\lim_{x\to 0} \frac{\sin^2 3x}{x^3 + x^2}$	9 January 8, 1994
4	Find the limit , if it exists	$\lim_{x\to 0} \frac{\sin^2 x + x^2}{x \sin 3x}$	16 November 2, 1996
5	Find the limit , if it exists	$\lim_{x\to 0} \frac{2x + 2\tan 3x}{\sin 2x}$	38 March 31, 2004
6	Evaluate the following limit(if it exists)	$\lim_{x\to 0} \frac{2x + \sin x}{\tan x}$	26 March 25, 1999
7	Evaluate the following limit (if it exists)	$\lim_{x\to 0} \frac{x + 2\sin(x)}{\tan(3x) - x}$	49 July 5, 2008
8	Find	$\lim_{x\to 0} \frac{x + \sin x}{3x - \tan(2x)}$	50 November 17, 2008 A
9	Find the limit , if it exists	$\lim_{x\to 0} \frac{x^2 + x \sin x}{x \tan 2x}$	25 April 11, 1999
10	Find the limit , if it exists)	$\lim_{x\to 0} \frac{x \tan 2x}{3x^2 + 2\sin^2 x}$	47 November 10.2007 A
11	Find the limit , if it exists	$\lim_{x\to 0} \frac{x^2 - x \sin x + \tan^2 x}{x^2}$	

Homework

12	Find the limit , if it exists	$\lim_{x\to\frac{\pi}{2}} \sin(x-\cos x)$	
13	Evaluate the following limit (if it exists)	$\lim_{x\to 0} \frac{\tan^2 5x}{x^3+x^2}$	
14	Find	$\lim_{x\to 0} \frac{4x+2\tan x}{6x-\sin 2x}$	22 July 18, 1998 A
15	Find the limit , if it exists	$\lim_{x\to 0} \frac{1-\cos 2x}{x\tan 3x}$	12 January 9, 1995
16	Evaluate the following limit (if it exists)	$\lim_{x\to 0} \frac{\sin 6x}{x\cos 3x}$	31 June 5, 2008
17	Evaluate the following limit (if it exists)	$\lim_{x\to 0} \frac{\sec 3x\tan 3x}{5x}$	35 August 15, 2009
18	Evaluate each of the following limits, if it exists:	$\lim_{x\to 0} \frac{\sin x}{\sqrt{x^3+2x^2}}$	56 July 10, 2010
19	[2 pts.] Find	$\lim_{x\to 2} \frac{x\sin(x-2)}{x^2-3x+2}$	39 5 June, 2011
20	Find the following limit , if it exists	$\lim_{x\to 0} \frac{\lvert\sin x\rvert}{x}$	14 January 1996
21	Evaluate the following limit	$\lim_{x\to 0} \frac{2x+\sin 3x}{x-\tan 6x}$	40 October 28, 2004 A

20	Find the following limit , if it exists $\lim_{x\to 0}\frac{\lvert\sin x\rvert}{x}$

Solution

$$L=\lim_{x\to 0}\frac{\lvert\sin x\rvert}{x}$$

$$\lim_{x\to 0^+}\frac{\lvert\sin x\rvert}{x}=\lim_{x\to 0^+}\frac{\sin x}{x}=1$$

$$\lim_{x\to 0^-}\frac{\lvert\sin x\rvert}{x}=\lim_{x\to 0^-}\frac{-\sin x}{x}=-1$$

$\therefore L$ $D.N.E$

21	Evaluate the following limit $\lim_{x\to 0}\frac{2x+\sin 3x}{x-\tan 6x}$

Solution

$$L=\lim_{x\to 0}\frac{2x+\sin 3x}{x-\tan 6x}=\lim_{x\to 0}\frac{2+\frac{\sin 3x}{x}}{1-\frac{\tan 6x}{x}}=\lim_{x\to 0}\frac{2+3\cdot\frac{\sin 3x}{3x}}{1-6\cdot\frac{\tan 6x}{6x}}=\frac{2+3(1)}{1-6(1)}=\frac{2+3}{1-6}=\frac{5}{-5}=-1$$